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Approximations of Zeros of Entire Functions by Zeros of Polynomials

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Approximations of entire functions by polynomials are considered. Residual bounds for zeros of entire functions are derived. © 2000 Academic Press

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1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Consider the entire function

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{a_k \lambda^k}{(k!)^\rho} \qquad (\lambda \in \mathbf{C},\ a_0 = 1,\ \rho > 1/2) \tag{1.1}$$

with complex, in general, coefficients. Assume that

$$C_f \equiv \sum_{k=0}^{\infty} |a_k|^2 < \infty \tag{1.2}$$

and put

$$p_n(\lambda) \equiv \sum_{k=0}^{n} \frac{a_k \lambda^k}{(k!)^\rho}, \qquad \text{and} \qquad q_n \equiv \left[\sum_{k=n+1}^{\infty} |a_k|^2 \right]^{1/2} \quad (n < \infty).$$

Our main problem is: If q_n is small, how close are the zeros of p_n to those of f? The variation of the zeros of general analytic functions under perturbations was investigated, in particular, by P. Rosenbloom [7]. He established the perturbation result that provides the existence of a zero of a perturbed function in a given domain. In the present paper a new approach to

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the problem is proposed. It is based on recent estimates for the norm of the resolvent of a Hilbert–Schmidt operator.

Note that due to the Schwarz inequality, relation (1.2) implies

$$|f(\lambda)|^2 \leqslant C_f \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{(k!)^{2\rho}} \leqslant C_f e^{|\lambda|^2}.$$

In order to formulate the result set

$$w_n = \zeta(2\rho) - 1 + \sum_{k=1}^{n} |a_k|^2 - |z_k(p_n)|^{-2},$$

where $\zeta(.)$ is the Riemann Zeta function, $z_k(p_n)$ $(k = 1, 2, ..., n)$ are the zeros of p_n taken with their multiplicities. Since $a_0 = 1$, $z_k(p_n) \neq 0$ $(k = 1, 2, ..., n)$.

The aim of the present paper is to prove the following

THEOREM 1.1. *Under condition* (1.2), *all the zeros of f are in the set*

$$\bigcup_{j=0}^{n} \Omega_j,$$

where

$$\Omega_0 = \{z \in \mathbf{C} : q_n \sqrt{2}\ |z| \exp[w_n\ |z|^2] \geqslant 1\}$$

and

$$\Omega_j = \left\{ z \in \mathbf{C} : q_n \sqrt{2}\ |z_j^{-1}(p_n) - z^{-1}|^{-1} \exp\left[\frac{w_n}{|z_j^{-1}(p_n) - z^{-1}|^2}\right] \geqslant 1 \right\}$$

$$(j = 1, ..., n).$$

All the proofs are presented in the next section.

Let now a_k, $k = 1, 2, ...,$ be real. Then as it is proven below, the inequality

$$w_n \leqslant v(p_n) \equiv \sum_{k=2}^{n} a_k^2 + \zeta(2\rho) - 1 + a_2 2^{1-\rho} \tag{1.3}$$

is valid. So in the case of real coefficients *we can replace w_n everywhere below by the easily calculated quantity $v(p_n)$.*

We also will prove the following

THEOREM 1.2. *Let condition* (1.2) *be fulfilled. In addition, let* $\psi(q_n, w_n)$ *be the unique positive* (*simple*) *root of the equation*

$$q_n \sqrt{2}\, x^{-1} \exp\left[\frac{w_n}{x^2}\right] = 1. \tag{1.4}$$

Then any zero $z(f)$ *of* f *either satisfies the inequality*

$$|z(f)| \geqslant \frac{1}{\psi(q_n, w_n)}, \tag{1.5}$$

or there is a zero $z(p_n)$ *of* p_n, *such that*

$$|z(p_n) - z(f)| \leqslant \psi(q_n, w_n)\, |z(p_n)\, z(f)|. \tag{1.6}$$

Further, relation (1.6) yields

$$|z(p_n) - z(f)| \leqslant \psi(q_n, w_n)(|z(p_n) - z(f)| + |z(p_n)|)\, |z(p_n)|.$$

Consequently,

$$|z(p_n) - z(f)|\, (1 - \psi(q_n, w_n)\, |z(p_n)|) \leqslant \psi(q_n, w_n)\, |z(p_n)|^2.$$

Hence, we get

COROLLARY 1.3. *Under the conditions* (1.2) *and*

$$\psi(q_n, w_n) \max_{k=1, \ldots, n} |z_k(p_n)| < 1, \tag{1.7}$$

any $z(f)$ *either satisfies the inequality* $|z(f)| > 1/\psi(q_n, w_n)$, *or there is a* $z(p_n)$, *such that*

$$|z(p_n) - z(f)| \leqslant \frac{\psi(q_n, w_n)\, |z(p_n)|^2}{1 - \psi(q_n, w_n)\, |z(p_n)|} \leqslant \frac{|z(p_n)|}{1 - \psi(q_n, w_n)\, |z(p_n)|}. \tag{1.8}$$

Furthermore, relation (1.6) gives

$$|z(p_n)| - |z(f)| \leqslant \psi(q_n, w_n)\, |z(f)|\, |z(p_n)|.$$

Hence,

$$|z(f)| \geqslant (\psi(q_n, w_n)\, |z(p_n)| + 1)^{-1}\, |z(p_n)|.$$

This inequality yields the following result

COROLLARY 1.4. *For a positive number R, let the relation*

$$R_1 \equiv \frac{R}{\psi(q_n, w_n) \min_K |z_k(p_n)| + 1} < \psi^{-1}(q_n, w_n)$$

hold. In addition, let p_n have no zeros in the circle $\Omega(R) = \{z \in \mathbf{C} : |z| \leqslant R\}$. Then f has no zeros in the ring

$$\{z \in \mathbf{C} : R_1 \leqslant |z| \leqslant \psi^{-1}(q_n, w_n)\}.$$

Below we will prove the inequality

$$\psi(q_n, w_n) \leqslant \sqrt{2w_n \gamma_n^{-1}}, \tag{1.9}$$

where

$$\gamma_n = \ln(1/2 + \sqrt{1/4 + w_n/q_n^2}).$$

Due to (1.3) everywhere above *we can replace $\psi(q_n, w_n)$ by the easily calculated quantity $\sqrt{2v(p_n)\gamma_n^{-1}}$, if p_n is real.*

2. PROOFS

Consider the entire function

$$h(\lambda) = \sum_{k=0}^{\infty} \frac{b_k \lambda^k}{(k!)^\rho} \quad (\lambda \in \mathbf{C}, b_0 = 1, \rho > 1/2)$$

with complex, in general, coefficients, under the assumption

$$C_h \equiv \sum_{k=0}^{\infty} |b_k|^2 < \infty. \tag{2.1}$$

Put

$$q \equiv \left[\sum_{k=1}^{\infty} |a_k - b_k|^2 \right]^{1/2}.$$

Let l^2 be the Hilbert space of number sequences with the norm

$$\|x\| = \left[\sum_{k=1}^{\infty} |x_k|^2 \right]^{1/2} \qquad (x = (x_k) \in l^2).$$

Introduce in l^2 operators A and B by virtue of the infinite matrices

$$A = \begin{pmatrix} -a_1 & -a_2 & -a_3 & -a_4 & \cdots \\ 1/2^\rho & 0 & 0 & 0 & \cdots \\ 0 & 1/3^\rho & 0 & 0 & \cdots \\ 0 & 0 & 1/4^\rho & 0 & \cdots \\ . & . & . & . & \cdots \end{pmatrix}$$

and

$$A = \begin{pmatrix} -b_1 & -b_2 & -b_3 & -b_4 & \cdots \\ 1/2^\rho & 0 & 0 & 0 & \cdots \\ 0 & 1/2^\rho & 0 & 0 & \cdots \\ 0 & 0 & 1/4^\rho & 0 & \cdots \\ . & . & . & . & \cdots \end{pmatrix}. \tag{2.2}$$

In addition, consider the $n \times n$ matrix

$$A_n = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1/2^\rho & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1/3^\rho & 0 & \cdots & 0 & 0 \\ & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1/n^\rho & 0 \\ . & & & & & \end{pmatrix}.$$

The direct calculations show that $p_n(\lambda) = \det(U_n - \lambda A_n)$, where I_n is the unit $n \times n$-matrix. So the eigenvalues $\lambda_k(A_n)$ of A_n taken with their multiplicities, satisfy the relations $\lambda_k(A_n) = z_k^{-1}(p_n)$ $(k = 1, ..., n)$. In other words

$$\sigma(A_n) = \{\mu \in \mathbf{C} : \mu = z_k^{-1}(p_n), k = 1, ..., n\}.$$

Here and below $\sigma(A)$ denotes the spectrum of an operator A. Denote by $\tilde{A}_n$ the operator in l^2 presented by matrix A_n. That is, $\tilde{A}_n = A_n \oplus 0$. Clearly, operators $\tilde{A}_n$ converge in the operator norm of l^2 to A and

$$\sigma(\tilde{A}_n) = \sigma(A_n) \cup 0 = \{\mu \in \mathbf{C} : \mu = z_k^{-1}(p_n), k = 1, ..., n\} \cup 0.$$

Thanks to the continuity of isolated eigenvalues (Kato [4, p. 213]), for any finite k we have

$$\lambda_k(A_n) \to \lambda_k(A) \in \sigma(A) \qquad \text{as} \quad n \to \infty.$$

Due to the continuous dependence of zeros of entire functions on its coefficients, $z_k(p_n) \to z_k(f)$ as $n \to \infty$. Therefore,

$$\sigma(A) = \{\mu \in \mathbf{C} : \mu = z_k^{-1}(f), k = 1, 2, ...\} \cup 0.$$

Similarly,

$$\sigma(B) = \{\mu \in \mathbf{C} : \mu = z_k^{-1}(h), k = 1, 2, ..., \} \cup 0,$$

where $z_k(h)$, $k = 1, 2, ...,$ are the zeros of h with their multiplicities.

Furthermore, it is easy to see that $\|A - B\| \leqslant q$. Let I be the unit operator in l^2. Obviously,

$$\begin{aligned}(I - zA)^{-1} &- (I - zB)^{-1} \\ &= z(I - zA)^{-1} (A - B)(I - zB)^{-1} \; (z^{-1} \notin \sigma(A) \cup \sigma(B)).\end{aligned}$$

Hence,

$$\begin{aligned}\|(I - zA)^{-1}\| &\leqslant \|(I - zB)^{-1}\| + q\,|z|\, \|(I - zB)^{-1}\|\, \|(I - zA)^{-1}\| \\ &= \|(I - zB)^{-1}\| + q\, \|(z^{-1}I - B)^{-1}\|\, \|(I - zA)^{-1}\|.\end{aligned}$$

So, if

$$q\, \|(z^{-1}I - B)^{-1}\| < 1,$$

then

$$\|(I - zA)^{-1}\| \leqslant \|(I - zB)^{-1}\|\, (1 - q\, \|(z^{-1}I - B)^{-1}\|)^{-1}.$$

Consequently, the operator

$$(Iz^{-1} - A)^{-1} = z(I - zA)^{-1}$$

is bounded and thus $z^{-1} \notin \sigma(A)$. Hence, it follows that for any zero $z(f) \neq \infty$, one can write

$$q\, \|(z^{-1}(f)\, I - B)^{-1}\| \geqslant 1. \tag{2.3}$$

Furthermore, assume that for any regular λ,

$$\|(\lambda I - B)^{-1}\| \leqslant G(\rho^{-1}(B, \lambda)), \tag{2.4}$$

where $G(x)$ $(x \geqslant 0)$ is a continuous scalar-valued function positive and increasing on $[0, \infty)$ with the property $G(0) = 0$, and $\rho(B, \lambda)$ is the distance between λ and $\sigma(B)$. If the set of all zeros of h is infinite, then by (2.4) for any regular point of B, there is a zero $z(h)$, such that

$$\|(\lambda I - B)^{-1}\| \leqslant G(|\lambda - z^{-1}(h)|^{-1}).$$

If the set of all zeros of h is finite with $\beta(h) \equiv \max_k |z_k(h)|$, then $\lambda = 0$ is an isolated point of the spectrum of B and under the inequality $|\lambda| \leqslant 1/(2\beta(h))$, we have $\rho(B, \lambda) = |\lambda|$. Thus by virtue of relation (2.4), it can be written

$$\|(\lambda I - B)^{-1}\| \leqslant G(|\lambda|^{-1}).$$

Now according to (2.3) we get the following result.

LEMMA 2.1. *Under conditions* (1.2), (2.1), (2.4) *any* $z(f)$ *either satisfies the inequality*

$$G(|z(f)|)g \geqslant 1, \tag{2.5}$$

or there is a $z(h)$, *such that*

$$qG(|z^{-1}(h) - z^{-1}(f)|^{-1}) \geqslant 1. \tag{2.6}$$

COROLLARY 2.2. *Under conditions* (1.2), (2.1), (2.4), *let* $r(q)$ *be the unique positive root of the scaler equation*

$$qG(1/y) = 1. \tag{2.7}$$

Then any $z(f)$, *either satisfies the inequality* $|z(f)| \geqslant r^{-1}(q)$, *or there is* $z(h)$, *such that*

$$|z^{-1}(h) - z^{-1}(f)| \leqslant r(q).$$

Indeed, since G increases, this result follows from (2.5), (2.6), and (2.7).

LEMMA 2.3. *Under conditions* (1.2), (2.1), *and* (2.4) *any zero* $z(f)$ *of* f, *either satisfies the inequality*

$$\sqrt{2}\, q\, |z(f)| \exp[w(h)\, |z(f)|^2] \geqslant 1,$$

or there is a zero $z(h)$ *of* h, *such that*

$$\sqrt{2}\, q\, |z^{-1}(h) - z^{-1}(f)|^{-1} \exp[w(h)\, |z^{-1}(h) - z^{-1}(f)|^{-2}] \geqslant 1,$$

where

$$w(h) = \sum_{k=1}^{\infty} |b_k|^2 - |z_k^{-1}(h)|^2 + \zeta(2\rho) - 1.$$

Proof. We apply the following result (Gil' [1, p. 53]). Let B be an arbitrary Hilbert–Schmidt operator. Then

$$\|(B - I\lambda)^{-1}\| \leqslant \sum_{k=0}^{\infty} \frac{g^k(B)}{\sqrt{k!}\, \rho^{k+1}(B, \lambda)} \qquad \text{for all regular} \quad \lambda,$$

where

$$g(B) = \left(N^2(B) - \sum_{k=1}^{\infty} |\lambda_k(B)|^2 \right)^{1/2}.$$

Here $\lambda_k(B)$, $k = 1, 2, ...,$ are the eigenvalues of B with their multiplicities and $N(B)$ is the Hilbert–Schmidt norm of B, i.e., $N^2(B) = \mathrm{Trace}(BB^*)$. By the Schwarz inequality,

$$\left[\sum_{k=0}^{\infty} \frac{g^k(B)}{\sqrt{k!}\, \rho^{k+1}(B, \lambda)} \right]^2$$

$$= \left[\sum_{k=0}^{\infty} \frac{g^k(B)(\sqrt{2})^k}{(\sqrt{2})^k \sqrt{k!}\, \rho^{k+1}(B, \lambda)} \right]^2 \leqslant 2\rho^{-2}(B, \lambda) \exp\left[\frac{2g^k(B)}{\rho^2(B, \lambda)} \right].$$

Thus, for an arbitrary Hilbert–Schmidt operator B,

$$\|(B - I\lambda)^{-1}\| \leqslant \sqrt{2}\, \rho^{-1}(B, \lambda) \exp\left[\frac{g^2(B)}{\rho^2(B, \lambda)} \right] \tag{2.8}$$

In the considered case (2.2), we have

$$N^2(B) = \sum_{k=1}^{\infty} |b_k|^2 + k^{-2\rho} - 1 < \infty.$$

Therefore

$$g^2(B) = \zeta(2\rho) - 1 + \sum_{k=1}^{\infty} |b_k|^2 - |z_k^{-1}(h)|^2 = w(h).$$

Thus (2.8) yields

$$\|(B-\lambda I)^{-1}\| \leqslant \sqrt{2}\, \rho^{-1}(B, \lambda) \exp\left[\frac{w(h)}{\rho^2(B, \lambda)}\right].$$

Now the required result is due to Lemma 2.1. ▮

Proof of Theorem 1.1. Put $b_k=0$ for $k>n$ and $b_k=a_k$ for $k\leqslant n$. Then $h(z)=p_n(z)$, $q=q_n$ and $w(h)=w_n$. Now Lemma 2.3 implies the required result. ▮

Proof of Inequality (1.3). Let the coefficient of p_n be real. Since

$$\sum_{k=1}^{n} |z_k^{-1}(p_n)|^2=\sum_{k=1}^{n} |\lambda_k(B)|^2 \geqslant \sum_{k=1}^{n} \lambda_k^2(B)=\text{Trace } B^2$$

and

$$\text{Trace } B^2=a_1^2-a_2 2^{1-\rho},$$

it can be written

$$w(p_n)=g^2(B)\leqslant \zeta(2\rho)-1+\sum_{k=1}^{n} a_k^2-a_1^2+a_2 2^{1-\rho}$$

$$=\sum_{k=2}^{n} a_k^2+\zeta(2\rho)-1+a_2 2^{1-\rho}=v(p_n),$$

as claimed. ▮

The assertion of Theorem 1.2 follows from Corollary 2.2 and inequality (2.8).

Proof of Inequality (1.9). Equation (1.4) is equivalent to

$$2q_n^2 x^{-2} \exp\left[\frac{2w_n}{x^2}\right]=1.$$

Substitute in (1.4) the equality $y=2w_n x^{-2}$. Then

$$\frac{w_n}{q_n^2}=ye^y.$$

Since $z \leqslant e^z - 1 (z \geqslant 0)$, the relation

$$\frac{w_n}{q_n^2} \leqslant e^{2y} - e^y$$

holds. Consequently, $e^y \geqslant r_{1,2}$, where $r_{1,2}$ are the roots of the polynomial

$$z^2 - z - \frac{w_n}{q_n^2}.$$

Hence, $y \geqslant \gamma_n$. This proves inequality (1.9). ∎

3. EXAMPLE

Consider the function

$$f(z) = c_0 + c_1 z + c_2 z^2 + l_1 e^{-zh_1} + l_2 e^{-zh_2} \qquad (0 \leqslant h_1, h_2 = \text{const} < 1)$$

with real coefficients c_0, c_1, c_2, l_1, l_2. As it is well known, such quasipolynomials play an essential role in the theory of differential-difference equations; cf. Hale [3], Kolmanovskii and Nosov [6], and Kolmanovskii and Myshkis [5]. Usually, stability conditions for quasipolynomials are investigated. But for many applications, estimates for the zeros of quasipolynomials are very important, e.g., Gil' [2, Chap. 9] and references therein. Theorem 1.1 allows us to derive estimates for the roots of quasipolynomials.

Without any loss of generality, assume that $c_0 + l_1 + l_2 = 1$. We have

$$f(z) = 1 + (c_1 - l_1 h_1 - l_2 h_2)\, z + z^2 (2c_2 + l_1 h_1^2 + l_2 h_2^2)/2$$
$$+ \sum_{k=3}^{\infty} z^k [l_1(-h_1)^k + l_2(-h_2)^k](k!)^{-1}. \tag{3.1}$$

Rewrite this function in the form (1.1) with $\rho = 1$, and

$$a_k = (-1)^k \, [l_1 h_1^k + l_2 h_2^k] \quad (k \geqslant 3), \qquad a_1 = c_1 - l_1 h_1 - l_2 h_2,$$
$$a_2 = 2c_2 + l_1 h_1^2 + l_2 h_2^2.$$

Put

$$p_2(\lambda) = 1 + a_1 z + a_2 z^2/2. \tag{3.2}$$

We have

$$q_2^2 = \sum_{k=3}^{\infty} l_1^2 h_1^{2k} + l_2 h_2^{2k} = l_1^2 h_1^6 (1-h_1^2)^{-1} + l_2^2 h_2^6 (1-h_2^2)^{-1}$$

and $v(p_2) = a_2^2 + a_2 + \zeta(2) - 1$. So due to Theorem 1.1 and relation (1.3), we can assert that all the zeros of f are in the set $\bigcup_{j=0}^{2} \Omega_j$, where

$$\Omega_0 = \{ z \in \mathbf{C} : q_2 \sqrt{2}\, |z| \exp[v(p_2)\, |z|^2] \geqslant 1 \}$$

and

$$\Omega_j = \left\{ z \in \mathbf{C} : q_2 \sqrt{2}\, |z_j^{-1}(p_2) - z^{-1}|^{-1} \exp\left[\frac{v(p_2)}{|z_j^{-1}(p_2) - z^{-1}|^2} \right] \geqslant 1 \right\} \qquad (j = 1, 2).$$

Besides, $z_1(p_2)$, $z_2(p_2)$ are the roots of polynomial (3.2).

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